

The Mathematics Hidden in a Sheet of Paper

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Paper Formats

The international standard for paper sizes is defined by ISO 216 and includes two main series. Series A includes formats A0 to A10, while series B includes formats B0 to B10. This standard is based on the original DIN 476 norm, which has been in use since 1922 in Germany. It was developed by the German mathematician and physicist Walter Porstmann.

The two series share two fundamental properties:

1. All formats are similar rectangles.
2. A smaller format is created from a larger one by halving it, i.e. dividing it into two mutually symmetric rectangles.¹

These properties are not arbitrary. They have both aesthetic value and practical applications. For example, each sheet of paper in the system can be produced from the largest sheet simply by cutting, without generating any waste.

Each series also has a specific additional property:

- In the A series, the area of the largest paper A0 is 1 m^2 .
- In the B series, the shorter side of the largest format B0 measures 1, m.

Exercise 1. Determine the similarity ratio (reduction factor) between two consecutive paper formats, and also find the ratio of adjacent sides that each format must satisfy.

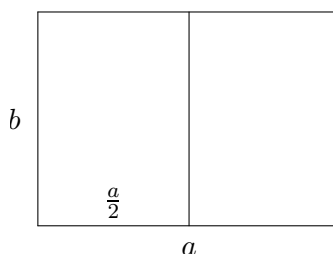


Figure 1: Notation of side lengths for Exercise 1

¹The side lengths of the formats that were created by halving are rounded down to whole millimeters. The most commonly used A4 format has dimensions $210 \times 297 \text{ mm}$.

Exercise 2. Calculate the dimensions of the largest A0 paper format, assuming that the lengths of its sides are integers in millimetres and its area is as close as possible to one square metre.

Exercise 3. In addition to the properties shared with series A, series B also has the property that the shorter side of the largest format B0 measures exactly one metre. Show that if we assume that format A0 has an area of exactly one square metre, and we allow all formats to have non-integer dimensions, then for every non-negative integer n , the following holds:

$$S(B(n+1)) = \sqrt{S(A(n)) \cdot S(A(n+1))},$$

i.e. the area of format $B(n+1)$ is the geometric mean of the areas of formats $A(n)$ and $A(n+1)$.

Paper Folding

You may have once wondered how many times a sheet of A4 paper can be folded in half—and maybe you even tried it yourself. But what probably didn't occur to you is that a mathematician can answer this question without ever folding the paper.

Let us consider the following simple model of paper folding.

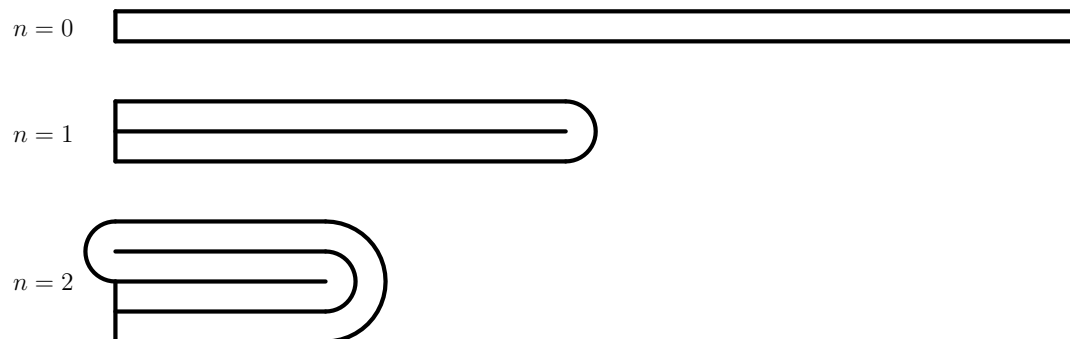


Figure 2: Paper folding model

Each time the paper is folded in half, part of it is used to create the fold. We can model the shape of the fold as a semicircle whose radius equals the thickness of the paper. We can also observe that paper stacks up with each fold: we start with a single layer, after the first fold we have two layers, after the second fold four layers, and so on. In the following problems, we will work with this model.

Exercise 4. What would be the thickness of the stacked office paper after four, seven, ten, twenty-one, and forty-two folds? Assume that the thickness of the paper is $t_0 = 0.1$ mm.

These results show that there must be some limit to how many times a sheet of paper can be folded. One way to understand this limit is by examining how much paper is actually consumed by the fold itself during each step.

Exercise 5. How much paper is “lost” during folding?

Exercise 6. How many times can a typical A4 office paper with a thickness of 0.1mm be folded?

As a curiosity, let us add that the first person to derive the equation from Exercise 5 was a high school student, Britney Gallivan from California. She is currently the Guinness World Record holder for the most times a piece of paper has been folded in half. She managed to fold paper twelve times. However, she couldn't use standard A4 paper to achieve this—she used toilet paper with a length of 1.219 metres. Moreover, she employed a different folding technique, alternating the folding directions.

Literature

1. Niss, Mogens; Bluem Werner. *The Learning and Teaching of Mathematical Modelling*, Routledge 2020, 978-1-315-18931-4
2. *Most times to fold a piece of paper*. <https://www.guinnessworldrecords.com/world-records/494571-most-times-to-fold-a-piece-of-paper>
3. *Wikipedia. Paper size*. https://en.wikipedia.org/wiki/Paper_size