Math4You

2023–2025

Mathematics hidden in a sheet of paper

Paper formats

The international standard for paper sizes is given by ISO 216 and includes two basic series. Series A includes formats A0-A10 and series B includes formats B0-B10. This standard is based on the original DIN 476 standard, which was used in Germany since 1922. It was created by the German mathematician and physicist Walter Porstmann.

Both series have two common basic properties:

1. All formats are similar rectangles.
2. The smaller format is created by halving the larger one, i.e. dividing it into two mutually axially symmetrical rectangles.[1]

These properties are not chosen at random. They have both aesthetic significance and practical uses. For example, each sheet of paper in a given system can be made from the largest piece by simply cutting it, and no waste is created.

Series A and B each have a certain special property:

- In series A, the area of the largest paper A0 is $1\,\mathrm{m}^2$.
- In series B, the largest format B0 has a shorter side of length $1\,\mathrm{m}$.

Exercise 1. Determine the similarity coefficient (reduction) of two consecutive paper formats and also determine the ratio of adjacent sides that each of the formats must adhere to.

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[1] The side lengths of the formats that were created by halving are rounded down to whole millimeters. The most commonly used A4 format has dimensions $210 \times 297\,\mathrm{mm}$.

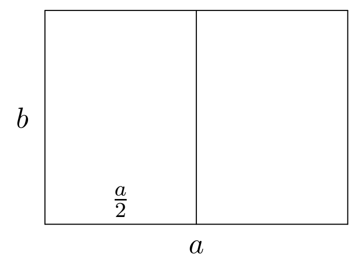

Figure 1: Notation of side lengths for Exercise 1

Solution. First, let's realize that a rectangle cannot be a square because no bisecting a square can produce a square. We bisect the rectangle in question along the axis of the longer side of this rectangle. If we bisect along the axis of the shorter side, we would not get a rectangle similar to the original – the longer side will not change and the shorter side will be shortened.

If we denote a the longer side of the rectangle, b its shorter side, and k the reduction coefficient of two consecutive formats, we have $k \cdot a = b$ and $k \cdot b = \frac{a}{2}$. Substituting the first expression of b into the second equation, we get

$$k^2 \cdot a = \frac{a}{2} \quad / : a$$

$$k^2 = \frac{1}{2} \qquad \rightarrow \qquad k = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

From the equation $k \cdot a = b$ it follows that the ratio of the sides of the rectangle $a : b$ is the reciprocal of the coefficient k, i.e. $\sqrt{2}$.

Exercise 2. Calculate the dimensions of the largest A0 format if you know that its side lengths are integers in mm and its area is as close as possible to one square meter.

Solution. From the previous problem, we know that the dimensions of an A0 sheet are b_0 (shorter side) and $b_0 \cdot \sqrt{2}$ (longer side) for the unknown length b_0 that needs to be calculated. We know that

$$b_0 \cdot b_0 \cdot \sqrt{2} = 1000000 \,\text{mm}^2,$$

and therefore, after expressing b_0 and rounding the result to the ones place (to nearest whole number), we get the value $b_0 \doteq 841\,\text{mm}$. The length of the longer side of an A0 sheet is then the product of $841 \cdot \sqrt{2} \doteq 1189\,\text{mm}$.

Exercise 3. The series of formats B, in addition to the common properties valid for both series A and B, also has the property that the length of the shorter side of the largest format B0 is equal to one meter. Show that if format A0 has an area of exactly one square meter and if we assume non-integer dimensions for all formats, then for every non-negative integer n the following relation holds

$$S(\mathsf{B}(n+1)) = \sqrt{S(\mathsf{A}(n)) \cdot S(\mathsf{A}(n+1))},$$

i.e. the area of format B$(n+1)$ is the geometric mean of the areas of formats A(n) and A$(n+1)$.

Solution. Since the shorter side of the B0 format measures 1 m, its longer side measures $\sqrt{2}$ m according to the solution of Exercise 1 (also valid for the B format, because we are based on the same properties). Thus, the content of the B0 format is $\sqrt{2}\,\text{m}^2$ and each subsequent sheet of the B(n) format has half the content of the previous one, thus $S\left(\mathsf{B}(n)\right) = \frac{\sqrt{2}}{2^n}\,\text{m}^2$ for each non-negative integer n.

Since further $S(\mathsf{A}0) = 1\,\text{m}^2$ and each subsequent sheet of the format A(n) has half the content of the previous one, then $S\left(\mathsf{A}(n)\right) = \left(\frac{1}{2}\right)^n = \frac{1}{2^n}\,\mathsf{m}^2$ for each n. Thus

$$\begin{aligned}\sqrt{S(\mathsf{A}(n)) \cdot S(\mathsf{A}(n+1))} &= \sqrt{\frac{1}{2^n} \cdot \frac{1}{2^{n+1}}} = \sqrt{\frac{1}{2^n} \cdot \frac{1}{2^n} \cdot \frac{1}{2}} \\ &= \frac{1}{2^n} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2^{n+1}} = S(\mathsf{B}(n+1)).\end{aligned}$$

Paper folding

You may have wondered how many times you can fold an A4 sheet of paper in half, and you may have even tried it yourself. But you probably didn't think that a mathematician could answer this question without having to fold the paper at all.

Let's imagine the following simple model of folding paper.

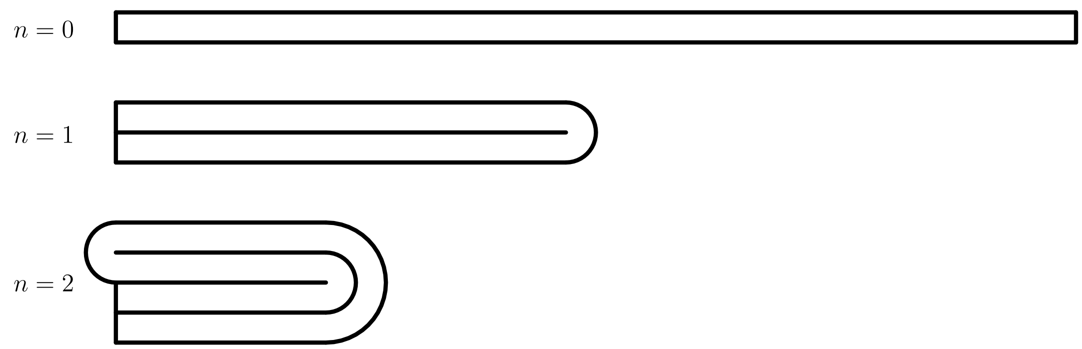

Figure 2: Paper folding model

When folding paper in half, we always use up some of the paper to create the fold. We can model its shape as half a circle, the radius of which is equal to the thickness of the paper. In addition, we can also notice that the paper becomes layered when folded. At the beginning, we have only one layer, after the first fold, two layers, after the second fold, four layers, etc. In the following tasks, we will work with this model.

Exercise 4. What would be the thickness of the layered office paper after four, seven, ten, twenty-one, and forty-two folds? Let's assume that the thickness of our sheet of paper is $t_0 = 0{,}1$ mm.

Solution. It is easy to see that after k folds we get a total of 2^k layers of paper. The thicknesses would be

$$\begin{aligned} t_4 &= t_0 \cdot 2^4 = 1{,}6\,\mathsf{mm} \\ t_7 &= t_0 \cdot 2^7 = 12{,}8\,\mathsf{mm} \\ t_{10} &= t_0 \cdot 2^{10} = 102{,}4\,\mathsf{mm} \\ t_{21} &= t_0 \cdot 2^{21} \approx 209{,}7\,\mathsf{m} \\ t_{42} &= t_0 \cdot 2^{42} \approx 439\,804\,\mathsf{km} \end{aligned}$$

According to the results of the previous problem, it can be seen that there must be a limit to the paper folding. One way to know this limit is by examining how much paper is actually lost in the folding process.

Exercise 5. How much paper is "lost" in the folding process?

Solution. Consider a paper of thickness t. The first fold creates a semicircle of radius t (see previous figure), so we need πt of paper to fold. The second fold creates two semicircles. One with radius t and the other with radius $2t$, so we need $\pi t + 2\pi t$ of paper and together

$$\pi t + (\pi t + 2\pi t)\,.$$

The third fold creates a semicircle with radii t, $2t$, $3t$ and $4t$. We therefore lose $\pi t + 2\pi t + 3\pi t + 4\pi t$ of paper. The total loss will be

$$\pi t + (\pi t + 2\pi t) + (\pi t + 2\pi t + 3\pi t + 4\pi t)$$

Analogously, after n compositions we lose

$$\pi t + (\pi t + 2\pi t) + \cdots + (\pi t + 2\pi t + \cdots + 2^{n-1}\pi t)$$

of paper. If we subtract πt, we can notice that we have in parentheses the sum of the first terms of the arithmetic sequence

$$\pi t\left[1 + (1+2) + (1+2+3+4) + \cdots + (1+2+\cdots+2^{n-1})\right]\,.$$

If we use the repeated formula for the sum of the first terms of an arithmetic sequence, we get

$$\frac{\pi t}{2}(1\cdot 2 + 2\cdot 3 + 4\cdot 5 + \cdots + 2^{n-1}\cdot(2^{n-1}+1))\,.$$

Here, the k-th term can generally be written as

$$2^{k-1}\cdot\left(2^{k-1}+1\right) = (2^2)^{k-1} + 2^{k-1}.$$

The relation for the total paper loss can therefore be rewritten as

$$\frac{\pi t}{2}\left[\left((2^2)^0 + (2^2)^1 + \cdots + (2^2)^{n-1}\right) + \left(2^0 + 2^1 + \cdots + 2^{n-1}\right)\right]\,.$$

We thus get the sum of the first terms of two geometric sequences, so we can use the formula for their sum and get

$$\frac{\pi t}{2}\left(\frac{2^{2n}-1}{3} + 2^n - 1\right)\,.$$

After removing $\frac{1}{3}$ from the parentheses, we have

$$\frac{\pi t}{6}\left((2^n)^2 + 3\cdot 2^n - 4\right)$$

and by factoring it into a product, we get

$$\frac{\pi t}{6}(2^n+4)(2^n-1)\,.$$

This last formula actually expresses a kind of estimate of the minimum length of paper of thickness t that we need in order to fold it n times.

Exercise 6. How many times can a typical A4 office paper with a thickness of 0.1mm be folded?

Solution. Using the result of the previous problem, we know that we are looking for the largest natural number n such that

$$\frac{\pi \cdot 0{,}1}{6}(2^n + 4)(2^n - 1) < 297.$$

An exact solution to this inequality would not be easy, but fortunately it is not necessary. We just need to substitute suitable n:

$$\frac{\pi \cdot 0{,}1}{6}(2^6 + 4)(2^6 - 1) \doteq 224{,}31;$$
$$\frac{\pi \cdot 0{,}1}{6}(2^7 + 4)(2^7 - 1) \doteq 877{,}76.$$

According to this model, it is possible to fold a piece of paper of a given size at most six times.

For your interest, the first person to derive the equation for Exercise 5 was Britney Gallivan, a high school student from California. She currently holds the Guinness World Record for folding a piece of paper in half the most number of times. She has folded the paper twelve times in total. However, she could not use normal A4 paper to achieve this. Instead, she used $1\,219$ metres of toilet paper. She also used a different folding technique, alternating the directions.

Literature

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