Math4You

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Which lottery ticket is more profitable?

Very often in life we find ourselves in situations where chance and probability are involved. Imagine being faced with a choice between several options - for example, when choosing a lottery ticket or investing in a project. Each choice has its risks and potential rewards, but the question is, how to find out which one is the most advantageous? This is where the so-called *expected value* comes into play.

Expected value tells us what outcome we can expect on average if we choose a particular option. It helps us better estimate what will pay off in the long run. It is not an exact prediction, but a tool that allows us to better understand risk and reward, both in simple games and in real life decisions.

For example, consider two lottery numbers:

- Lot A: It costs 10 CZK and with probability $0{,}1$ we win 100 CZK, otherwise we win nothing.
- Lot B: It costs 10 CZK and with probability $0{,}2$ we win 60 CZK, otherwise we win nothing.

For ticket A, we expect that if we buy 10 tickets, one of them will win 100 CZK and nine will win nothing. So we can expect that each lottery ticket will bring us 10 CZK on average.

Similarly, for lottery ticket B, we expect that if we buy 10 tickets, two of them will win 60 CZK and eight will win nothing. We can therefore expect each lottery ticket to bring us 12 CZK on average.

Therefore the lottery ticket B is more profitable.

Expected value

The average win we just calculated is called the *expected value* (or also the *mean value*).

In general, we can say that for a random variable X that takes on finitely many values $x_1, \ldots, x_k$ with probabilities $p_1, \ldots, p_k$, we calculate its expected value

$$EV = \sum_{i=1}^{k} x_i p_i.$$

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Which lottery ticket is the best?

Let's take a look at three lottery tickets. A black pearl worth 50 CZK, the Black Pearl worth 100 CZK and the Rentiér lottery ticket worth 50 CZK.

The prize structure for the 50 CZK Black Pearl lottery tickets, of which there are $13,000,000 in total, is as follows as follows.

Amount of money won on the lottery ticket (in CZK)	Number of winning tickets
50	1 820 000
100	1 040 000
150	260 000
200	130 000
300	130 000
500	104 000
1 000	5 550
2 000	2 300
4 000	480
10 000	185
20 000	84
100 000	14
1 500 000	6
Celkem	3 492 619

The prize structure for the 100 CZK Black Pearl lottery ticket looks similar, with a total of $15,000,000 issued.

Amount of money won on the lottery ticket (in CZK)	Number of winning tickets
100	2 400 000
200	900 000
300	450 000
500	150 000
600	150 000
900	75 000
1 000	75 000
1 500	20 000
6 000	4 000
20 000	185
50 000	84
100 000	30
200 000	13
5 000 000	6
Celkem	4 224 318

For the 3rd and final round, let's take a look at the Rentiér lottery ticket, of which $8,000,000 has been issued and the prizes are shown in the table below.

Amount of money won on the lottery ticket (in CZK)	Number of winning tickets
50	960 000
100	720 000
150	160 000
250	160 000
500	70 000
1 000	1 300
2 000	500
5 000	160
10 000	80
100 000	6
3 500 000	3
Celkem	2 072 049

The top prize of $3\,500\,000$ Kč is not paid at once, but consists of an immediate prize of $500\,000$ Kč and an annuity of $50\,000$ Kč for 5 years.

Exercise 1. Which ticket has the highest chance of winning?

Solution. In the case of the Black Pearl ticket for 50 CZK, out of the total number of 13,000,000$ pieces, there are $3\,492\,619$ winning tickets (see the last row of the table). The probability that a randomly selected ticket is a winning one can be calculated as

$$P(V_1) = \frac{3\,492\,619}{13\,000\,000} = 0{,}268633\,.$$

We can say that if we buy one lottery ticket, we have a chance of winning about $26{,}86\,\%$. By adjusting the fraction, we can also find that the chance of getting a winning lottery ticket is $1:3{,}72$.

Similarly, in the case of the Black Pearl lottery ticket, worth $100, we get

$$P(V_2) = \frac{4\,224\,318}{15\,000\,000} = 0{,}2816212\,.$$

That is, the chance of winning is $28{,}16\,\%$ or $1:3{,}55$.

In the case of the Rentier ticket, we have

$$P(V_3) = \frac{2\,072\,049}{8\,000\,000} = 0{,}259\,,$$

so the chance of winning is $25{,}9$, or $1:3{,}86$.

Comparing the individual probabilities of winning, we see that the greatest chance of winning is when buying a Black Pearl ticket worth 100 CZK.

In this context, we can also consider what we call a winning ticket. A winning ticket is one that has a prize. But if we paid 100 CZK for the ticket, then a win of 100 CZK will pay us back, but we haven't actually won anything. In order to get the probability of actually winning, we won't consider the first row in our winning tables. This way we get adjusted winning probabilities

$$P(V_1) = \frac{1\,672\,619}{13\,000\,000} = 0{,}128633$$
$$P(V_2) = \frac{1\,824\,318}{15\,000\,000} = 0{,}1216212$$
$$P(V_3) = \frac{1\,112\,049}{8\,000\,000} = 0{,}139\,.$$

We can see that if we consider the lottery tickets that actually win more than their cost, the best lottery ticket is the Rentiér lottery ticket, where the chance of winning is $13{,}9\,\%$.

Exercise 2. What is the expected value of each ticket?

Řešení. To calculate the expected value, by definition, we need to know the probabilities of individual wins:

Amount won on the lottery ticket (in CZK)	Probability of a given winning
50	0,14
100	0,08
150	0,02
200	0,01
300	0,01
500	0,008
1 000	0,000 426 9
2 000	0,000 176 923
4 000	0,000 036 923
10 000	0,000 014 231
20 000	0,000 006 461 538
100 000	0,000 006 461 538
1 500 000	0,000 000 461 538

If we denote the values of individual winnings n_1 to n_{13} and their corresponding probabilities p_1 to p_{13}, we get the expected value $EV(L_1)$ of the Black Pearl ticket

$$EV(L_1) = \sum_{k=1}^{13} n_k p_k = 29\,\text{Kč}.$$

Given how the individual probabilities are calculated, we can also calculate the expected value as follows

$$EV(L_1) = \frac{1}{13\,000\,000}\left(50 \cdot 1\,820\,000 + 100 \cdot 1\,040\,000 + \cdots + 100\,000 \cdot 14 + 1\,500\,000 \cdot 6\right).$$

This approach is preferable as we do not have to calculate the probability of each possible win in the win table. For the 100 CZK worth Black Pearl ticket, we get the expected value $EV(L_2)$:

$$EV(L_2) = \frac{1}{15\,000\,000}\left(100 \cdot 2\,400\,000 + 200 \cdot 900\,000 + \cdots + 200\,000 \cdot 13 + 5\,000\,000 \cdot 6\right) = 64\,\text{Kč}.$$

And for the Rentiér lottery ticket we get the expected value $EV(L_3)$:

$$EV(L_3) = \frac{1}{8\,000\,000}\left(50 \cdot 960\,000 + 100 \cdot 720\,000 + \cdots + 100\,000 \cdot 6 + 3\,500\,000 \cdot 3\right) = 29{,}25\,\text{Kč}.$$

Note.

- Usually, lotteries state the total amount of winnings and the number of tickets, the expected value is of course the ratio of these two numbers.
- The stated values are often even lower in reality, as tax is often paid on winnings.
- The same approach can be used to calculate the expected value of a pack of various trading card games (Pokémon, Lorcana, Magic the Gathering or sports cards).

Exercise 3. In the previous examples, we considered the main prize of the Rentiér lottery worth $3\,500\,000$ CZK. But is this really the real value of the prize, given that it is not paid out all at once?

Solution. The simple answer is that it is not.

It is important to remember that if we got the money immediately, we could save it or invest it somehow. To find out what the value of 50 000 CZK that we will receive in a month is, we can use a concept called *present value*. When using it, we ask ourselves how much money we would have to invest today to get the amount we want in a month (e.g. considered 50 000 CZK). And this value is then the so-called present value.

Suppose we could save the given amount P_0 for a month with a monthly interest rate of $0{,}5\,\%$. We would then get $P_1 = 1{,}005P_0$ per month. The present value is then the amount P_0, which we must deposit so that P_1 is $50\,000$ CZK, i.e.

$$P_0 = \frac{50\,000}{1{,}005} = 49\,751{,}24\,\text{Kč}.$$

If we want to determine the present value of an amount that we will receive in n months, we assume that we will keep the given amount deposited for the entire time. We then use compound interest and get the present value P_0 of the amount P_n that we will receive in n months as

$$P_0 = \frac{P_n}{1{,}005^n}.$$

Recall that the main prize of the Rentiér lottery consists of $500\,000$ CZK and thirty monthly payments of $50\,000$ CZK. Considering a monthly interest rate of $0{,}5\,\%$, the present value PV of these payments is

$$PV = \frac{50\,000}{1{,}005} + \frac{50\,000}{1{,}005^2} + \cdots + \frac{50\,000}{1{,}005^{29}} + \frac{50\,000}{1{,}005^{30}}\,.$$

We can notice that this is the sum of the terms of a geometric sequence and thus the calculation can be significantly shortened.

$$PV = \frac{50\,000}{1{,}005} \cdot \frac{1 - \left(\frac{1}{1{,}005}\right)^{30}}{1 - \frac{1}{1{,}005}} = 1\,389\,702{,}7\,\text{Kč}.$$

Therefore we assume that the value of the main prize is only $1\,889\,702{,}7$ Kč.

If we use this amount to calculate to expected value of the Rentier lottery, we get (nebo: If we adjust the calculation of the expected value of the Rentier lottery ticket using this amount, we get.)

$$EV(L_3) = 28{,}65\,\text{Kč}.$$

Note. Previous considerations were still very simplistic, as they did not include, for example, the effect of inflation.

Exercise 4. Based on the results of the previous tasks, choose the lottery ticket that is most advantageous.

Solution. Based on the previous tasks, we can compare the lottery tickets according to various criteria:

1. By probability of winning.

According to this criterion, the best ticket is the Černá perla ticket worth 100 CZK, which has a chance of winning $28{,}16\,\%$, then the Black Pearl ticket worth 50 CZK with a chance of $26{,}86\,\%$ and the worst is the Rentiér ticket with a chance of $25{,}9\,\%$.

2. By probability of actual winning.

If we consider the chance of winning more than we paid, we get the following other order. The best is the Rentiér ticket with a chance of winning $13{,}9\,\%$, then the Černá perla ticket worth 50 CZK with a chance of $12{,}86\,\%$ and the last is the Black Pearl ticket worth 100 CZK with a chance of winning $12{,}16\,\%$.

3. By expected value.

The expected value of a 50 CZK Black Pearl ticket is 29 CZK. On average we lose 21 CZK on one ticket. Similarly, the expected value of a 100 CZK Black Pearl ticket is 64 CZK. On average we lose 36 CZK. And in the case of a 50 CZK Rentiér ticket, the adjusted expected value is 28,65 CZK, so on average we lose 21,35 CZK.

We can say that (expectedly) all tickets are losing. But we can consider the 50 CZK Black Pearl ticket to be the best, as it is the least loss-making.

Literature

- Novák, J., *Střední hodnota v úlohách na střední škole*. Učitel matematiky, 2, JČMF, 2024.
- *Herní plán loterií SAZKA* [online] Dostupné z https://static.sazka.cz/kentico-media/sazka/media/content/herni-plany/hp-sazka-od-17-7-24-komplet-sazka.pdf, [cit. 1. 9. 2024]