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Math4You

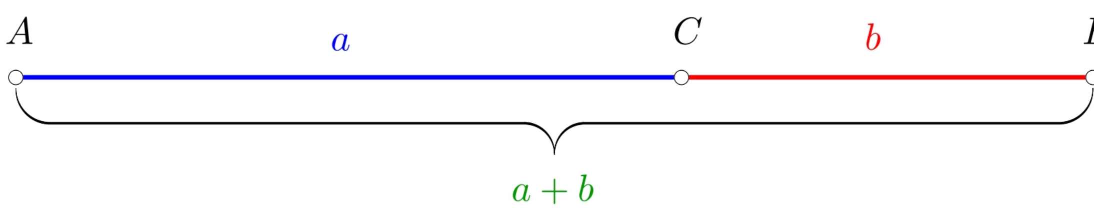
2023–2025

Golden Section and Chain Fraction

Let us have a segment AB and a point C on it. We say that the point C divides the line segment AB in the ratio of the golden section, if the relation

$$\frac{|AB|}{|AC|} = \frac{|AC|}{|CB|}$$

applies to the lengths of the considered segments. This ratio is often denoted by the Greek letter φ and has a value of approximately $1{,}618$.



$$\frac{a+b}{a} = \frac{a}{b}$$

Figure 1: A line segment divided in the ratio of the golden section

A nice example of the use of the golden ratio in everyday life is the credit card. It has the shape of a so-called golden rectangle, the sides of which meet the golden ratio. The golden rectangle is a popular shape because of its balanced appearance; it is neither too long nor too wide.

Co-funded by the Erasmus+ Programme of the European Union.

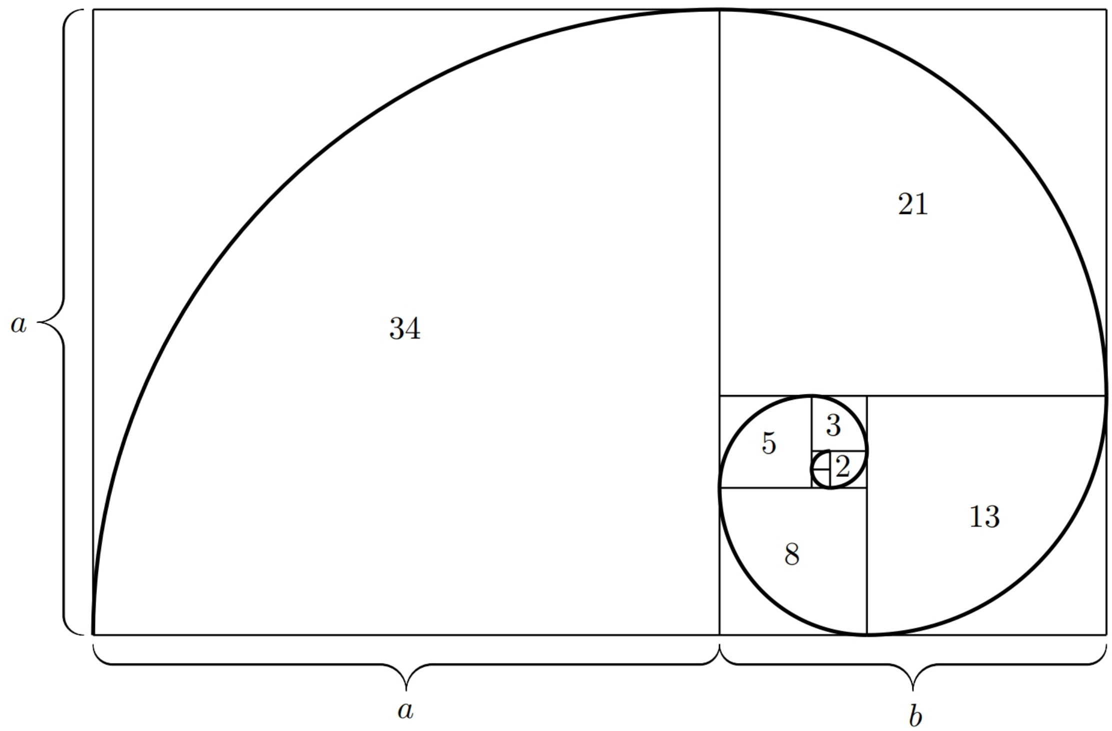


Figure 2: Golden rectangle and golden spiral

The golden ratio is closely related to the Fibonacci sequence. The members of the Fibonacci sequence are the numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ..., where each next member of the sequence is obtained by the sum of the previous two members. We also refer to the individual elements of this sequence as Fibonacci numbers. What is the connection between the Fibonacci sequence and the golden section? It is true that the limit of the ratios of two consecutive members of this sequence is exactly equal to the golden section φ.

If we construct squares whose side lengths correspond exactly to the Fibonacci numbers, it is possible to arrange them nicely next to each other in the shape of a golden rectangle as shown in the figure. We can then inscribe a quarter circle in each square and we get the so-called golden spiral. The golden spiral is a special case of the logarithmic spiral.

In nature, the golden ratio appears in the form of the Fibonacci sequence. We can find it in the arrangement of the leaves on the stems. The leaves grow one above the other so that they do not shade each other, the transition from one leaf to the next has the character of a helical ascent around the stem. Similar arrangements are found in the scales of the pine cone, the seeds of the sunflower, or the the pineapple peel. The logarithmic spiral is also found in the shells of molluscs or in the fiddlehead fern. Tornadoes, cyclones and galaxies also have this shape.

The golden section is widely used in art to achieve aesthetically impressive and harmonious compositions. Painters and photographers use this ratio to determine the placement of key elements in their works. Architects often integrate the golden section into building designs.

An infinite chain fraction/Continued fraction

An infinite chain fraction (also called continued fraction) is an expression of the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}},$$

where a_0 is an integer and the numbers a_i are positive natural numbers for $i \in \mathbb{N}$. A chain fraction can also be in finite form.

The golden section can be expressed by the continued fraction

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}.$$

Exercise 1. Calculate approximate values of the golden section using finite chain fractions

1. $$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}$$

2. $$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

Solution.

1. $$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} = 1 + \cfrac{1}{1 + \cfrac{1}{2}} = 1 + \frac{1}{\frac{3}{2}} = \frac{5}{3} \doteq 1{,}67$$

2. $$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}} = 1 + \cfrac{1}{1 + \cfrac{1}{\frac{3}{2}}} = 1 + \frac{1}{\frac{5}{3}} = \frac{8}{5} = 1{,}6$$

Exercise 2. Calculate the exact value of the golden ratio φ.

Solution. Assume that line segment AB has length 1. We divide this segment by the point C in the ratio of the golden section. Then it holds

$$\varphi = \frac{|AB|}{|AC|} = \frac{|AC|}{|CB|}.$$

Let's denote $x = |AC|$, i.e. x will be the length of the longer segment of the segment AB. Then $|BC| = 1 - x$ holds for the length of the line segment BC and thus we obtain the relation

$$\frac{1}{x} = \frac{x}{1-x}, \tag{1}$$

which makes sense for $x \neq 0$ a $x \neq 1$. However, we do not need to investigate these extreme values, because they certainly do not meet the golden ratio. By adjusting (1), we get a quadratic equation

$$x^2 + x - 1 = 0,$$

whose roots are

$$x_{1,2} = \frac{-1 \pm \sqrt{5}}{2}.$$

In our case, x is the length of the line segment; therefore, a negative value of x is meaningless. Thus, we have the only satisfying solution to equation (1)

$$x_1 = \frac{-1 + \sqrt{5}}{2}.$$

Now we can calculate the value of the golden section φ:

$$\varphi = \frac{|AB|}{|AC|} = \frac{1}{x} = \frac{1}{\frac{-1+\sqrt{5}}{2}} = \frac{2}{\sqrt{5}-1}.$$

By rationalizing the denominator we then get

$$\varphi = \frac{\sqrt{5}+1}{2} \doteq 1{,}618.$$

Exercise 3. Solve an equation inspired by the golden ratio in a finite chain fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{x}}}.$$

Solution.

First, we simplify the equation step by step.

$$
\begin{aligned}
x &= 1 + \cfrac{1}{1 + \cfrac{1}{\frac{x+1}{x}}} \qquad \text{pro } x \neq 0 \\
x &= 1 + \cfrac{1}{1 + \cfrac{x}{x+1}} \qquad \text{pro } x \neq -1 \\
x &= 1 + \cfrac{1}{\frac{x+1+x}{x+1}} \\
x &= 1 + \frac{x+1}{2x+1} \\
x &= \frac{3x+2}{2x+1}
\end{aligned}
$$

Under the condition $x \neq -\frac{1}{2}$ we obtain a quadratic equation from here

$$2x^2 - 2x - 2 = 0.$$

Her roots are

$$x_{1,2} = \frac{1 \pm \sqrt{5}}{2}.$$

Note that one of the solutions is again a value of the golden section.

Literature

- Wikipedia. *Golden ratio* [online]. Dostupné z https://en.wikipedia.org/wiki/Golden_ratio [cit. 10.,11.,2023].
- Wikipedia. *Řetězový zlomek* [online]. Dostupné z https://cs.wikipedia.org/wiki/Řetězový_zlomek [cit. 10.,11.,2023].