Twisted Prism

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Turning a Right Square Prism into a Twisted Prism

In furniture design, a shape appears that is also of interest from a geometrical point of view. It can be created from a right square prism by dividing each lateral face into two triangles using one of its diagonals and then rotating the top base by $90°$, as illustrated in Figure 1. While preserving the lengths of the edges and diagonals, the height of the prism changes. The resulting shape is a special case of a so-called *twisted prism*.

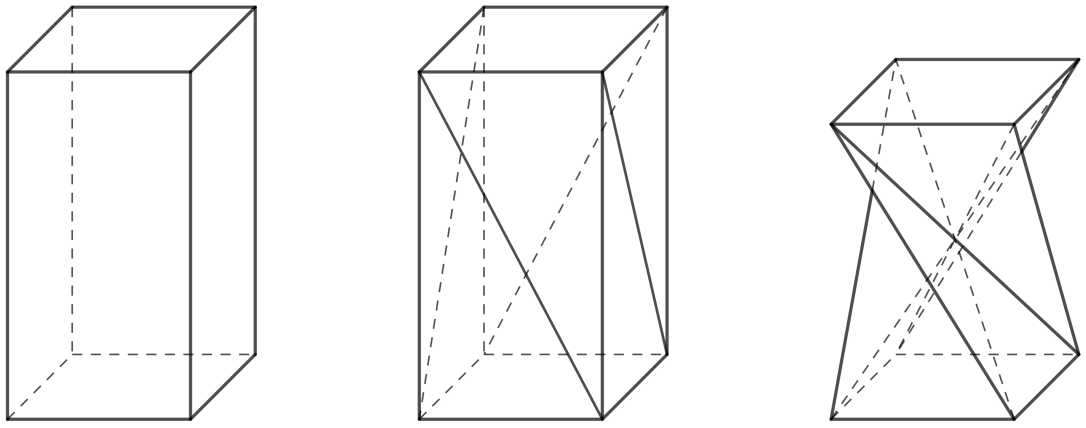

Figure 1: Transformation of a regular quadrilateral prism into its twisted variant while maintaining the lengths of the edges

Let's try creating this shape using, for example, a box from children's cough syrup or eye drops. For it to work properly, the box's lids (the top and bottom bases of the prism) must be square and able to be opened or unfolded in some way. Creating a twisted prism requires a bit of practice and skill, because performing a $90°$ twist is not so easy in reality.

We can use the following procedure:

- Open the box flaps and flatten the box into a single plane, so that two lateral faces are on the top and two on the bottom.
- Divide all lateral faces along diagonals in the same direction, as shown in Figure 2. Try to crease the paper along these lines so that it can be folded later.
- Gently fold each diagonal inward to form ridges.

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- Now, make the twist: hold the bottom part of the box lightly with one hand, lift the top part, and move it slightly to the left so that the top base rotates.
- Next, unfold the flattened box into a three-dimensional shape. This step takes a bit of practice. You can do it by pressing the box between two fingers at the point where all the marked diagonals overlap, while using your other hand to unfold the top or bottom part of the box into space.
- If you've made it this far, all that's left is to close the box flaps — and you're done. Congratulations!

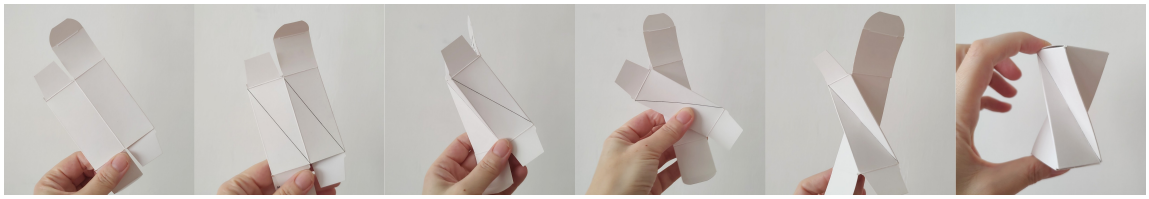

Figure 2: The process of making a twisted prism from a box

Exercise 1. We want to build a stool in the shape of a twisted prism, and we have several suitable cardboard boxes available. These boxes all have square bases with side length 40, cm, but their heights vary. From experience, we know that a stool height of 50, cm is comfortable for sitting. What box height is ideal for making a twisted rectangular prism stool with a vertical height of 50, cm? The side length of the base is $a = 40$, cm.

Other Variants of Twisted Prisms

Could a similar shape be created from right prisms whose base is a different regular n-gon?

The answer is yes. However, the construction method described above (for creating a twisted prism from paper) can only be applied when n is even.

In the following problem, we will focus on constructing a hexagonal twisted prism. First, try to figure out by how many degrees the top base is rotated in this case. If your spatial imagination fails you, build a model. To keep things simple, you can work with a net of the prism's lateral surface. In Figure 4 below, such a net is already prepared for assembly (thicker paper works best).

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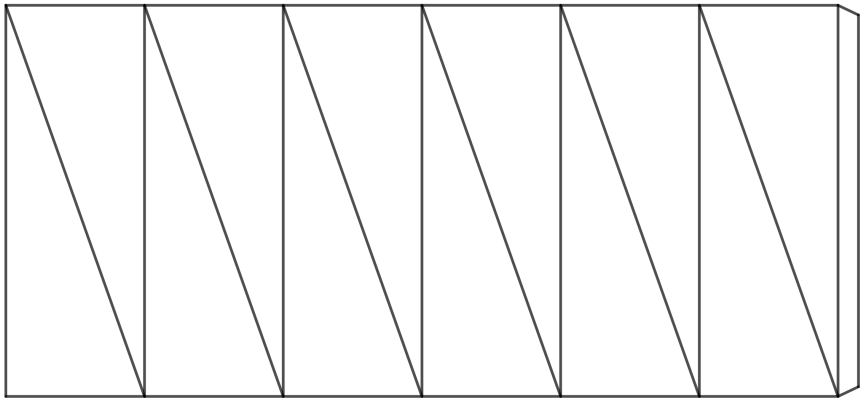

Figure 3: Net of the lateral surface of a hexagonal twisted prism

Before gluing, make folds along the edges and diagonals — fold the edges upward and the diagonals downward. After gluing, follow the instructions shown in Figure 5.

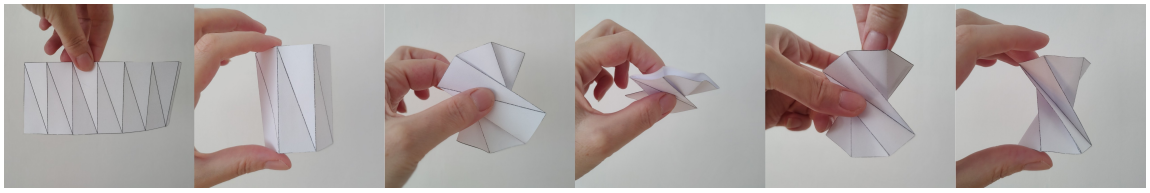

Figure 4: Steps for constructing a hexagonal twisted prism

Exercise 2. How does the height h of a twisted hexagonal prism (created from a regular hexagonal prism) depend on the original prism height v and the side length a of the base?

Exercise 3. What is the limiting condition for constructing the models in the previous two exercises?

Technique For Odd Values of nTechnique For Odd Values of n

It was already mentioned that the construction method described above does not work in the same way for odd values of n. However, the only difference is that once the net of the prism's lateral surface is

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prepared, it is not a good idea to glue the side edge before shaping the prism. Instead, first fold the net into the shape of a twisted prism, and only then glue the side edge.

Technique For Using a Different Rotation Angle

What if we wanted to make a shape similar to the one in the first example, but use it as a small vase for dried flowers or a pencil holder? In that case, we don't want the four edges of the twisted prism (formerly diagonals in the faces of the regular rectangular prism) to intersect in the middle — we want empty space inside. To achieve that, we need to reduce the rotation angle between the two bases.

Exercise 4. Construct a net of a vase in the shape of a twisted prism, given that the height of the vase is $h = 110,\text{mm}$, the side length of the square base is $a = 65,\text{mm}$, and the top base is rotated by $\alpha = 45°$ relative to the bottom base. This time, solve the problem constructively, using only a ruler and a compass.

Hint. The left diagram shows a top view of the bottom base and the top base rotated by $45°$. On the right, all edges of the resulting solid are also shown. You do not need to draw all of them — for the construction, the line segments AA' and BA' are the most important. It is also important to realize that in the top view, the orthogonal projection A_1 of point A' onto the plane of the bottom base satisfies $A_1 = A'$.

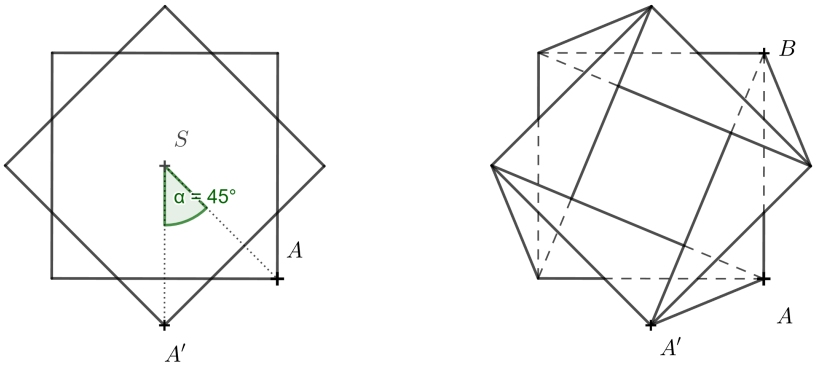

Figure 5: Top view of the vase

From the problem statement, we know that the height of the vase is $110,\text{mm}$, so $|AA_1| = 110,\text{mm}$. Using the right triangle A_1AA', we can determine the true length of edge AA'. Similarly, we can determine the length of edge BA' by constructing the right triangle A_1BA'. With this information, we know all the necessary edge lengths to draw the net of the vase.

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