Math4You

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The Curious Warehouse Keeper

When we solve purely mathematical problems, we get exact results. However, when we use mathematics to solve problems in the world around us, we rarely achieve absolute precision in the answer. Approximation is sometimes the result of a simplification of the real situation in our minds. Sometimes the input data are approximated (e.g. we can only measure lengths or time with limited accuracy) or an absolutely exact result is unrealistically unattainable and must be rounded off.

Rounding to a given number of valid digits is often used in practice (and in the following problems). We round a positive real number r to n valid digits as follows:

- We express r in the form $a \cdot 10^b$, where $a \in \mathbb{R}$,
- $a \in \langle 1, 10)$ and $b \in \mathbb{Z}$, and then we round the number a to $n-1$ decimal places according to the standard rules for rounding.
- E.g. numbers $r = 31{,}258\,16$ and $s = 0{,}023\,123\,6$ we round to four valid digits as follows:

$$r = 31{,}258\,16 = 3{,}125\,816 \cdot 10^1 \doteq 3{,}126 \cdot 10^1 = 31{,}26$$
$$s = 0{,}023\,123\,6 = 2{,}312\,36 \cdot 10^{-2} \doteq 2{,}312 \cdot 10^{-2} = 0{,}023\,12.$$

  In particular, rounding input data can have surprising consequences for the accuracy of the result, for example when solving equations, as we will see in the following series of problems.

A total of $401\,950$ CZK was paid for the delivery of 597 worth of Ixodinum vaccines against encephalitis and 386 worth of Nopolio vaccines against polio. However, upon initial inspection, 86 packs of Ixodinum vaccine and 19 packs of Nopolio vaccine were found to be expired and had to be returned. It was received $39\,600$ CZK for the returned medicines during the complaint process.

Exercise 1. The manager of the pharmaceutical warehouse received an invoice for two types of vaccine ordered. A total of $401\,950$ CZK was paid for the supply of 597 packages of Ixodinum vaccines against encephalitis and 386 packages of Nopolio vaccines against polio. However, upon initial inspection, 86 packs of Ixodinum vaccine and 19 packs of Nopolio vaccine were found to be expired and had to be returned. $39\,600$ CZK was received for the expired medicines during the complaint process.

Out of curiosity, the warehouse keeper wants to calculate the purchase price of a pack of both vaccines. However, he does not have a calculator or a mobile phone on hand, so he settles for an approximate solution. He rounds all the figures he knows to one valid digit before calculating.

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How much will his result differ from the actual purchase price? For both types of vaccine, determine the absolute difference between calculated and actual prices and the relative error in percent.

Solution. First, let's solve the problem without rounding. Let x be the price per pack of Ixodine and y be the price per pack of Nopolio. The information in the assignment leads to a system of two linear equations with two unknowns

$$\begin{aligned} 597x + \quad 386y &= 401\,950 \\ 86x + \quad 19y &= 39\,600, \end{aligned}$$

whose solution gives us the real purchase price of a package of Ixodin vaccine 350 CZK and the price of a package of Nopolio vaccine 500 CZK.

After rounding the coefficients to one valid digit, we solve the system

$$\begin{aligned} 600x' + \quad 400y' &= 400\,000 \\ 90x' + \quad 20y' &= 40\,000. \end{aligned}$$

The solution is the pair $x' = \frac{1000}{3} \doteq 333$ and $y' = 500$. Now we have the actual price of the medicine and the estimate of the price by the warehause keeper. We also calculate the relative error in the price of the medicine due to rounding. Relative error is the absolute error (absolute value of price difference) divided by the actual price per packagehe. We summarize the results in the table:

vaccine	actual price	price estimate	relative error
Ixodinum	350 Kč	333 Kč	$\frac{350-333}{350} = 4{,}9\,\%$
Nopolio	500 Kč	500 Kč	$\frac{500-500}{500} = 0\,\%$

Exercise 2. After a few months, another delivery arrived at the warehouse, namely a 504 packages of Antiflu vaccines against influenza and an 81 packages of Kontradift vaccines against diphtheria. $198\,900$ CZK was paid for this delivery. Upon entry inspection, 98 packs of Antiflu and 18 packs of Contradift were found to be expired. During their complaint, $40\,700$ CZK was returned.
The warehouse manager repeated his procedure and calculated the approximate purchase price of the two drugs by hand. This time, however, he was surprised. What was the reason for his surprise and how much did his result differ from the actual prices?

Solution. We will solve the problem in the same way as before, this time we will denote x the price of one pack of Antiflu and y the price of one pack of Contradift. The real prices are solution of the system

$$\begin{aligned} 504x + \quad 81y &= 198\,900 \\ 98x + \quad 18y &= 40\,700, \end{aligned}$$

where we get $x = 250$ and $y = 900$.

When rounding the coefficients, we solve the system

$$\begin{aligned} 500x' + \quad 80y' &= 200\,000 \\ 100x' + \quad 20y' &= 40\,000, \end{aligned}$$

whose solution is $x' = 400$ and $y' = 0$. From the solution of the warehouse manager, it seems that that the second vaccine was delivered to the warehouse free of charge, while it is actually 2 times more expensive than the first one. We calculate the relative error and enter all the values in the table again:

vaccine	actual price	price estimate	relative error
Antiflu	250 Kč	400 Kč	$\frac{400-250}{250} = 60\,\%$
Kontradift	900 Kč	0 Kč	$\frac{900-0}{900} = 100\,\%$

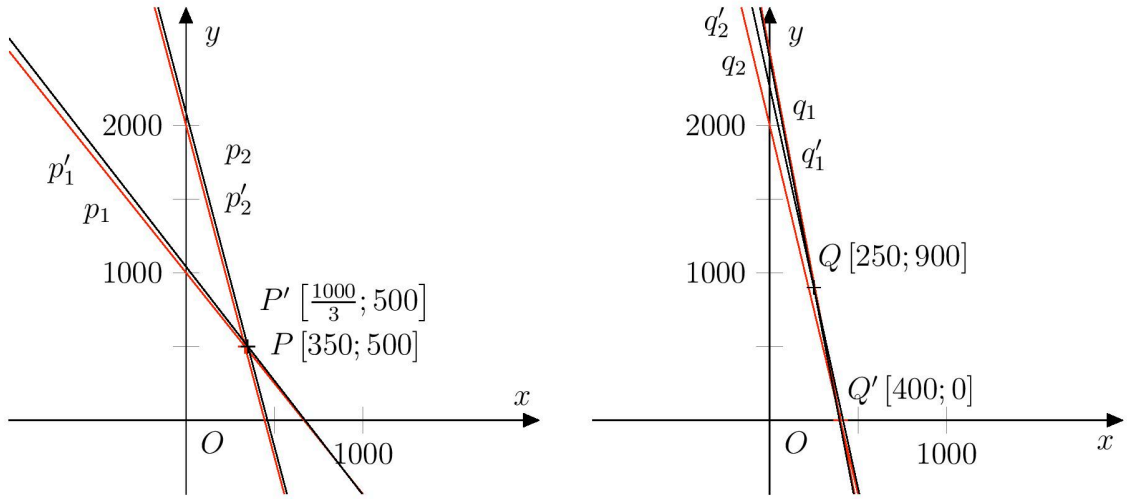
Exercise 3. Graphically represent the systems of equations from the previous two problems using appropriate software. By comparing the representation of the systems from Exercise 1 with the representation of the systems from Exercise 2, explain the difference in the accuracy of the results of the two exercises.

Solution. Let p_1, p_2 (or q_1, q_2) be the lines given by the equations of the system with unrounded coefficients in Exercise 1 (or Exercise 2), namely

$$p_1 \colon 597x + 386y = 401\,950$$
$$p_2 \colon 86x + 19y = 39\,600$$

$$q_1 \colon 504x + 81y = 198\,900$$
$$q_2 \colon 98x + 18y = 40\,700.$$

Let us denote the lines given by the corresponding equations with rounded coefficients by p_1', p_2', q_1' and q_2' and further denote the points $P \in p_1 \cap p_2$, $P' \in p_1' \cap p_2'$, $Q \in q_1 \cap q_2$ and $Q' \in q_1' \cap q_2'$. A graphical representation of the pair of systems for each problem separately is shown in following figure.



(a) Úloha 1 (b) Úloha 2

Figure 1: Graphic representation of the systems

By comparing the two graphical representations, it can be seen that in the case of Exercise 2, the pair of lines q_1 and q_2 are almost parallel. When rounding the coefficients of the equation, the position of the lines relative to the coordinate system generally changes and the position of the intersection also changes. The change in the position of the intersection is much greater for lines that are almost parallel. The figure also shows why the second coordinate of the intercept (i.e., the price of the vaccine Contradift) will be much more affected by rounding in the second problem. Because of the slope of the

lines q_1 and q_2, a small change in the x coordinate of the intersection would mean a large change in its y coordinate.

Literature

- Biermann K., Grötschel M., Lutz-Westphal B. (2010). *Besser als Mathe: Moderne angewandte Mathematik aus dem MATHEON zum Mitmachen*. Berlin: Vieweg+Teubner.