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Math4You

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The launch of the ball

Oblique launch

The oblique launch is the most general way of setting a object in a homogeneous gravitational field into motion. Suppose that a body (point mass) has been thrown obliquely in space without resistance. The initial velocity is $\vec{v}_0$ and the angle between and the vector $\vec{v}_0$ and horizontal direction is α. Introduce the Cartesian coordinate system with horizontal x-axis and vertical y-axis as on the picture. The coordinates of the initial velocity vector are

$$\vec{v}_0 = (v_0 \cos\alpha, v_0 \sin\alpha).$$

The motion of the body is governed by an acceleration of gravity g directed vertically downwards. The horizontal component of the gravitational acceleration is zero. Therefore the motion in the horizontal direction is unaffected by the gravitational field and the horizontal component of the motion is a motion with constant speed. The vertical component of the motion is affected by the negative acceleration $-g$ and it is a movement with constant acceleration and initial velocity $v_0 t \sin\alpha$ (an uniformly decelerated motion).

For the coordinates of the body we can use formulas for distance of motion with constant speed and constant acceleration an get

$$\begin{aligned} x(t) &= v_0 t \cos\alpha, \\ y(t) &= v_0 t \sin\alpha - \frac{1}{2} g t^2. \end{aligned} \tag{1}$$

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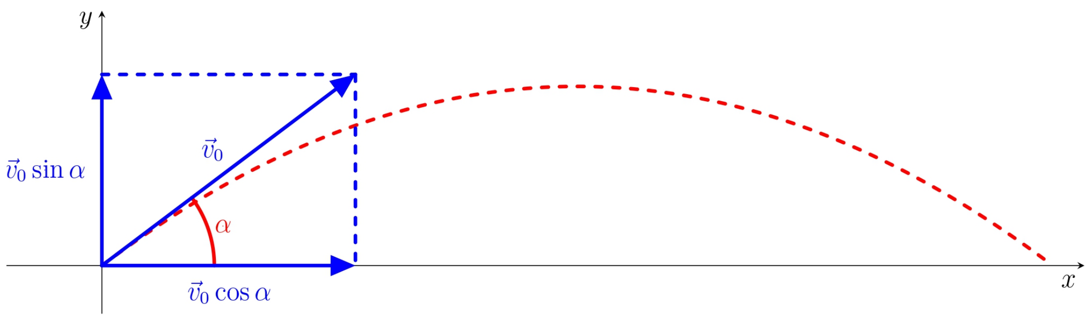

Figure 1: Šikmý vrh

Motion of golf ball

A golfer hits the ball with an initial velocity v_0. The angle between the initial velocity and the horizontal plane is α. Let's assume that drag forces are negligible. The motion of the ball therefore satisfies the conditions for movement of an obliquely thrown body in an environment without air resistance.

Exercise 1. Prove that the trajectory of a golf ball is a parabola.

Solution. To find the equation of the trajectory in the form $y = f(x)$ it is necessary to eliminate the parameter t from the system (1).

We solve the first equation with respect to time $t = \frac{x}{v_0 \cos\alpha}$ and substitute into the second equation:

$$y(x) = v_0 \sin\alpha \, \frac{x}{v_0 \cos\alpha} - \frac{1}{2} g \frac{x^2}{v_0^2 \cos^2\alpha} = -\frac{g}{2v_0^2 \cos^2\alpha} \cdot x^2 + \frac{\sin\alpha}{\cos\alpha} \cdot x \, .$$

From here we see that the y-coordinate of the trajectory is a quadratic function of the x-coordinate and the trajectory of the golf ball is therefore a parabola.

Exercise 2. Calculate the height of the throw, i.e. the maximal height y_{max} that the launched ball reaches.

Solution. The height of the throw is the maximum of the function from the previous exercise:

$$f\colon y = -\frac{g}{2v_0^2 \cos^2\alpha} \cdot x^2 + \frac{\sin\alpha}{\cos\alpha} \cdot x \, .$$

We calculate the derivative of the function f as

$$y' = -\frac{g}{2v_0^2 \cos^2\alpha} \cdot 2x + \frac{\sin\alpha}{\cos\alpha} \, .$$

To find the stationary point, we set the derivative equal to zero and obtain the equation

$$\frac{g}{v_0^2 \cos^2\alpha} \cdot x = \frac{\sin\alpha}{\cos\alpha} \, .$$

The solution of this equation is

$$x_{max} = \frac{v_0^2 \sin\alpha \cos\alpha}{g} \, .$$

Since the trajectory of motion is a concave down quadratic function, the located stationary point is a maximum and the vertical coordinate of this point is the height of the throw.

The height of the throw is calculated by evaluating the function f at the obtained coordinate x_{max}:

$$y_{max} = \frac{v_0^2 \sin^2 \alpha}{2g} \ .$$

Exercise 3. Given constant initial velocity, find the angle α which guaranties maximal distance between the initial and the terminal point on the trajectory.

Solution. To find the maximum range angle we need to obtain the terminal point x_d of the trajectory as a function of the angle α and find the maximum of the function $x_d(\alpha)$. Given that $y = 0$ when the ball hits a ground, we find zeros of the function

$$y(x) = -\frac{g}{2v_0^2 \cos^2 \alpha} \cdot x^2 + \frac{\sin \alpha}{\cos \alpha} \cdot x.$$

From here we obtain:

$$0 = -\frac{g}{2v_0^2 \cos^2 \alpha} \cdot x^2 + \frac{\sin \alpha}{\cos \alpha} \cdot x \ ,$$

$$0 = x \cdot \left(-\frac{g}{2v_0^2 \cos^2 \alpha} \cdot x + \frac{\sin \alpha}{\cos \alpha} \right) \ .$$

This factorized equation has two solutions. The first solution $x = 0$ corresponds to the place where the ball is launched and the second solution x_d to the place of impact

$$x_d(\alpha) = \frac{2v_0^2 \sin \alpha \cos \alpha}{g} = \frac{v_0^2}{g} \sin 2\alpha \ .$$

Now we need to find the maximum of the function $x_d(\alpha)$. It is sufficient to find the stationary point, since it has been showed that the trajectory is a parabola with vertex up. We calculate the derivative of the function $x_d(\alpha)$ by α

$$x_d'(\alpha) = \frac{v_0^2}{g} \cdot \cos 2\alpha \cdot 2 \ .$$

Letting the derivative equal to zero, we get $\cos 2\alpha = 0$, which is satisfied for $2\alpha = 90°$ (for the ball to be launched, obviously $\alpha \in \langle 0°, 90° \rangle$, so the solution is unambiguous). Thus the stationary point is $\alpha = 45°$.

The maximum range in golf is achieved when hitting the ball at an angle $\alpha = 45°$ and the ball lands at a distance of

$$x_d(45°) = \frac{v_0^2}{g} \sin(2 \cdot 45°) = \frac{v_0^2}{g}.$$

Note that it is possible to obtain the function $x_d(\alpha) = \frac{v_0^2}{g} \sin 2\alpha$ without calculus by using the symmetry of the parabola. The vertex of the parabola is located in the middle of the zeros. Therefore, for the x-coordinate of the impact point can be evaluated as $x_d(\alpha) = 2 \cdot x_{max}$. This avoids solving the quadratic equation obtained by substituting $y = 0$ into the function $y(x)$ and factorizing the right-hand side.

Literature

1. Kubera, Miroslav; Nečas, Tomáš; Beneš, Vojtěch. *Online učebnice fyziky pro gymnázia - Vrhy* [online]. Available from `https://e-manuel.cz/kapitoly/pouziti-pohybovych-zakonu/vyklad/vrhy/` [cit. 27.9.2023].
2. Moc, Ondřej; Eisenmann, Petr. *Šikmý vrh z rozhledny* [online]. Available from `https://mfi.upol.cz/files/26/2602/mfi_2602_129_137.pdf` [cit. 27.9.2023]