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Math4You

2023–2025

Distances on the Earth's surface

Which of the routes between Lisbon and Washington shown on the map is shorter?

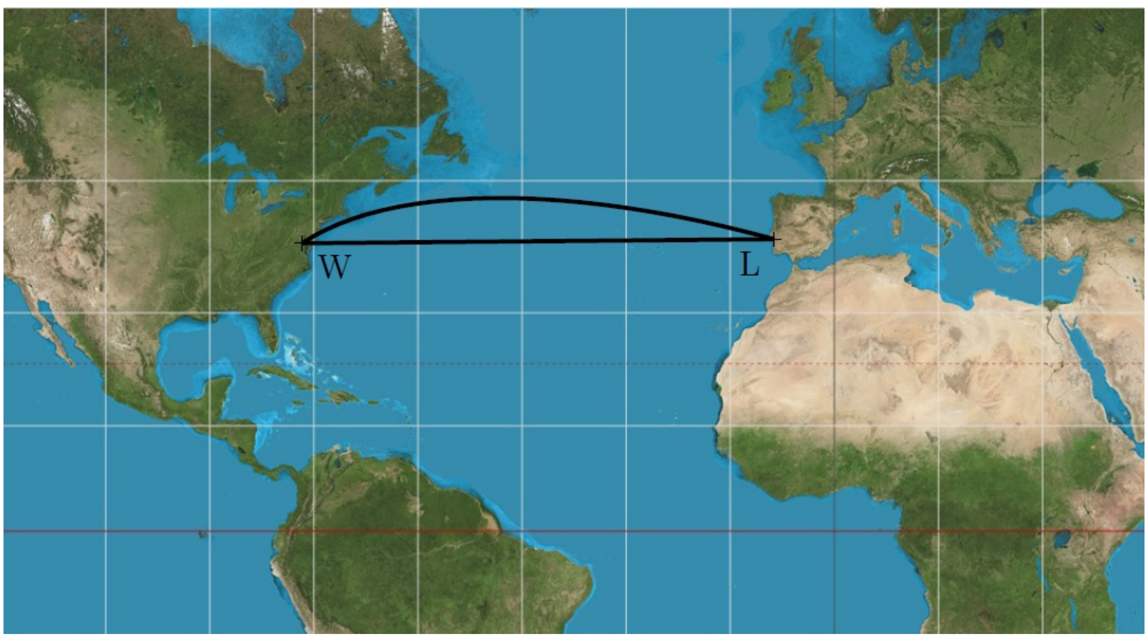

Figure 1: Map

This seemingly simple question has a surprising answer, as you will see in this exercise. The shorter route is the arc, the longer one is the line segment. The reason is the distortion of the distances in the selected representation of the Earth's surface. We see that the line segment LW on the map is approximately parallel to the geographic parallels on the Earth, so it actually corresponds to an arc on a circle that closely resembles a parallel (see the circle k with the center O in the figure).

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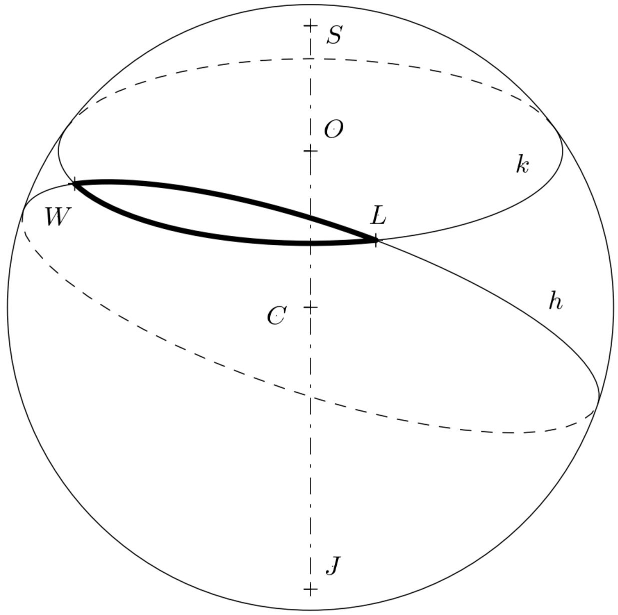

Figure 2: Parallel and great circle

However, on the spherical surface (which we will consider to be the Earth's surface in this task), the shortest distance is another arc. This arc lies on a circle h whose center C is the center of the Earth. We refer to such paths as *orthodromes* and call all circles with this property *great circles*. However, how many kilometers do we save by traveling along an orthodrome? The answer to this question has to be calculated.

Dictionary

- *Latitude* of a point on the Earth's surface (expressed in degrees and north/south orientation) is the angle between a straight line that passes through the given point and the center of the Earth and the plane of the equator.
- *Longitude* of a point on the Earth's surface (expressed in degrees and east/west orientation) is the angle between the plane of the meridian that passes through the given point and the plane of the zero meridian.

Exercise. Lisbon and Washington are located at approximately the same parallel (about 39° north latitude). How many kilometers does an airplane save by traveling on an orthodromic path compared to traveling on a parallel path? Lisbon is located at approximately 9° west longitude. Washington is located at 77° west longitude. Let us assume that the Earth is a sphere with center C and radius $6\ 371\,\text{km}$ and that the plane flies at an average altitude of $10\,\text{km}$ (take-off and landing are not taken into account). Therefore, in all considerations, we will work with a sphere of radius $\varrho = 6\ 381\,\text{km}$.

Solution. First, let's determine how many kilometers the plane will travel while traveling along the parallel. Let us denote the parallel at 39° north latitude as a circle k with center O and radius r. In a suitable rectangular projection of the globe (see the figure, where S and J are the poles), this circle appears as a segment AB with center O.

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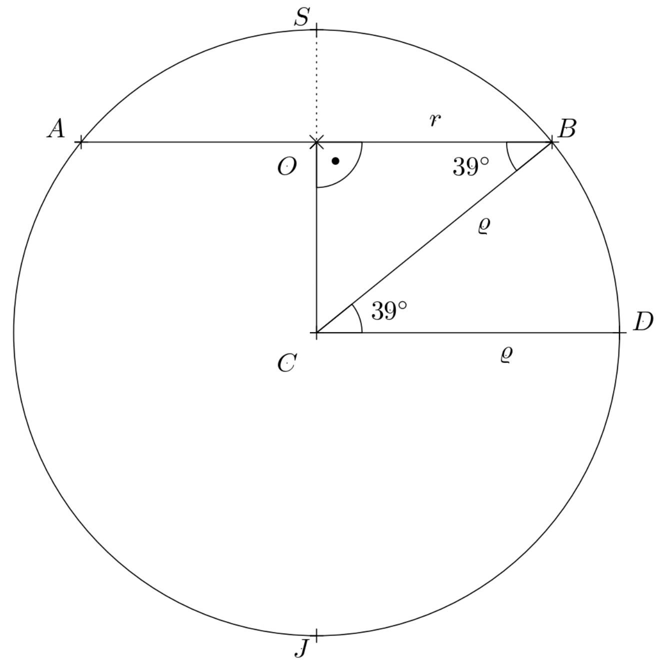

Figure 3: Rectangular projection

The figure shows the equality $|\sphericalangle CBO| = |\sphericalangle BCD|$ (the angles are alternating) and by using the cosine function in the rectangular triangle BSO we get $r = \varrho \cdot \cos 39°$.

The trajectory of a plane moving along a parallel (in the figure below, the plane's trajectory is represented by the shorter arc LW) is determined by direct proportion: the whole circle k has a length of $2\pi r = 2\pi\varrho \cdot \cos 39°$ km, i.e. the length of the shorter arc LW is equal to

$$\frac{(77-9)}{360} \cdot 2\pi\varrho \cdot \cos 39° \doteq 5\ 885{,}4\,\text{km}.$$

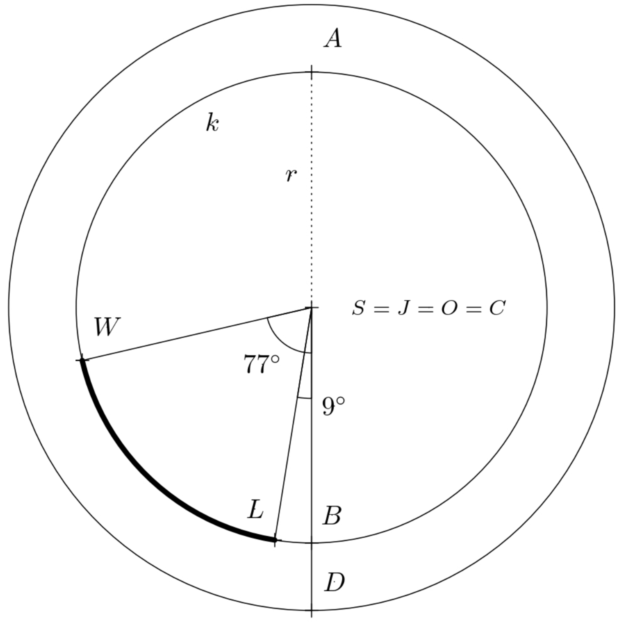

Figure 4: Rectangular projection - overlap of the poles.

Let us consider an isosceles triangle OWL, which we bisect with the height to the base LW into two congruent right triangles. In either of these two triangles, then $\frac{|LW|}{2} = r \cdot \sin 34°$ holds, and thus $|LW| = 2r \cdot \sin 34°$. If we make a similar consideration for the isosceles triangle CWL, we get the equality $|LW| = 2\varrho \cdot \sin \frac{\varphi}{2}$. By comparing the right sides of both derived equalities, we calculate the required angle φ:

$$2r \sin 34° = 2\varrho \sin \frac{\varphi}{2}$$

$$\sin \frac{\varphi}{2} = \frac{r \sin 34°}{\varrho} = \frac{\varrho \cos 39° \sin 34°}{\varrho} = \cos 39° \sin 34°$$

$$\frac{\varphi}{2} = \arcsin(\cos 39° \sin 34°) \doteq 25°45' \quad \Rightarrow \quad \varphi \doteq 51°30'.$$

We will determine the trajectory of the aircraft moving along the orthodrome similarly as in the case of a parallel line by direct proportion. The length of the whole circle h is equal to $2\pi\varrho$. Then, for the length of the shorter arc LW, the following applies

$$\frac{51{,}5}{360} \cdot 2\pi\varrho \doteq 5\ 735{,}5\,\text{km}.$$

We see that the two trajectories differ by approximately $150\,\text{km}$.

Literature

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- Hradecký F., Koman M., Vyšín J. *Několik úloh z geometrie jednoduchých těles.* Praha: Škola mladých matematiků. (1977). 36–38.

Sources of figures

- Mercator projection. Strebe – Vlastní dílo, CC BY-SA 4.0, dostupné z https://commons.wikimedia.org/wiki/File:Mercator_projection_Square.JPG [cit. 14. 8. 2023]