Fractal geometry

A fractal is an object whose geometric structure repeats within itself. The characteristic property of fractals is their self-similarity. Examples of fractals in nature include clouds, trees, or a cauliflower head. The word "fractal" comes from the Latin word "fractus", which translates broken or shattered. It was coined by Benoit B. Mandelbrot, considered the father of fractal geometry, renowned for his book The Fractal Geometry of Nature (1982).

In study of fractals, their dimension plays an important role. The topological dimension, known from classical Euclidean geometry, proved insufficient in describing fractals. Therefore, another type of dimension was needed. It was introduced by Felix Hausdorff, known as the Hausdorff dimension. For simple objects, we can understand it as the number:

$$d = \frac{\ln N}{\ln \frac{1}{r}},$$

where N is the number of parts the object is composed of, formed by self-similarity with the coefficient r from the original object. For example, for a square is true that it can be composed of four smaller squares that arise from it by self-similarity with the coefficient $r = \frac{1}{2}$, i.e.,

$$d = \frac{\ln 4}{\ln 2} = 2.$$

Thus, for a square, its fractal dimension (Hausdorff dimension) is the same as its normal intuitive dimension (topological dimension).

Koch snowflake

The *Koch snowflake* is a curve in the plane created by an iterative process from an equilateral triangle.

At the beginning, there is an equilateral triangle with sides of length 1. In each subsequent step, the following is performed:

1. Each line segment is divided into thirds.
2. An equilateral triangle is constructed above the middle third of the segment.
3. The base of the constructed triangle (formerly the middle third of the line segment) is removed.

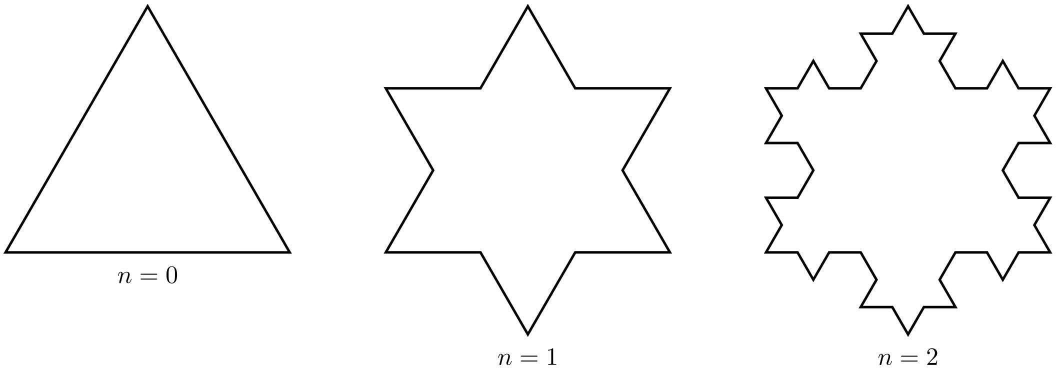

Figure 1: First iterations of Koch snowflake

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From the figure, we can see that to determine the length of one side of the snowflake after the first iteration, we need 4 sides of the triangle that was formed by reducing the side of the original triangle from the zero step with a self-similarity coefficient $r = \frac{1}{3}$, i.e.,

$$d = \frac{\ln 4}{\ln 3} \approx 1{,}26.$$

Since the Koch snowflake is a curve, we would expect its dimension to be 1. This discrepancy is due to the fact that the Koch snowflake is finally so fragmented that the resulting fractal has an infinite length but bounds a finite area plane structure.

Exercise 1. Calculate the perimeter of the Koch snowflake after the first, second, and third iterations.

Exercise 2. What is the perimeter of the Koch snowflake after the n-th iteration? Prove that the perimeter of the Koch snowflake is infinite.

Exercise 3. Calculate the area of the Koch snowflake after the first and second iterations.

Exercise 4. What is the area of the Koch snowflake after the n-th iteration? How many times larger is the area of the Koch snowflake relative to the original equilateral triangle?

Literature

- MathWorld. *Koch snowflake* [online]. Available from `https://mathworld.wolfram.com/KochSnowflake.html` [cit. 13. 7. 2023].
- *Koch curve* [online]. Available from `https://cs.wikipedia.org/wiki/Kochova_k%C5%99ivka` [cit. 13. 7. 2023].

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