Math4You

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Fractal geometry

A fractal is an object whose geometric structure repeats within itself. The characteristic property of fractals is their self-similarity. Examples of fractals in nature include clouds, trees, or a cauliflower head. The word "fractal" comes from the Latin word "fractus", which translates broken or shattered. It was coined by Benoit B. Mandelbrot, considered the father of fractal geometry, renowned for his book The Fractal Geometry of Nature (1982).

In study of fractals, their dimension plays an important role. The topological dimension, known from classical Euclidean geometry, proved insufficient in describing fractals. Therefore, another type of dimension was needed. It was introduced by Felix Hausdorff, known as the Hausdorff dimension. For simple objects, we can understand it as the number:

$$d = \frac{\ln N}{\ln \frac{1}{r}},$$

where N is the number of parts the object is composed of, formed by self-similarity with the coefficient r from the original object. For example, for a square is true that it can be composed of four smaller squares that arise from it by self-similarity with the coefficient $r = \frac{1}{2}$, i.e.,

$$d = \frac{\ln 4}{\ln 2} = 2.$$

Thus, for a square, its fractal dimension (Hausdorff dimension) is the same as its normal intuitive dimension (topological dimension).

Koch snowflake

The *Koch snowflake* is a curve in the plane created by an iterative process from an equilateral triangle.

At the beginning, there is an equilateral triangle with sides of length 1. In each subsequent step, the following is performed:

1. Each line segment is divided into thirds.
2. An equilateral triangle is constructed above the middle third of the segment.
3. The base of the constructed triangle (formerly the middle third of the line segment) is removed.

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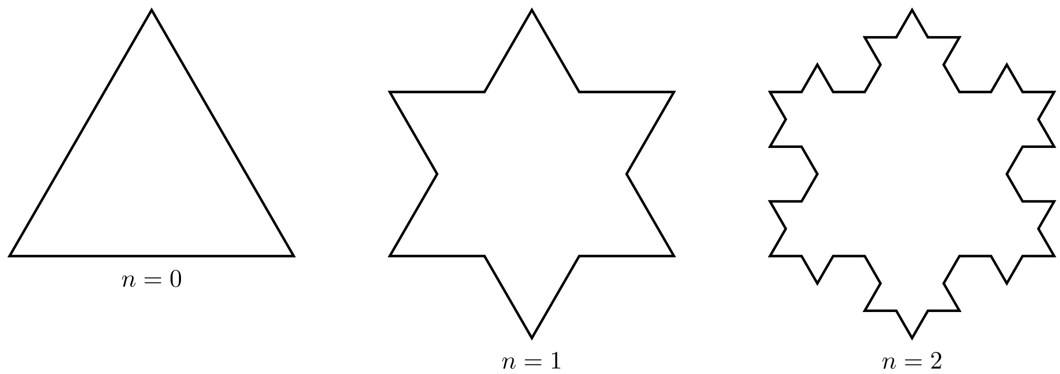

Figure 1: First iterations of Koch snowflake

From the figure, we can see that to determine the length of one side of the snowflake after the first iteration, we need 4 sides of the triangle that was formed by reducing the side of the original triangle from the zero step with a self-similarity coefficient $r = \frac{1}{3}$, i.e.,

$$d = \frac{\ln 4}{\ln 3} \approx 1{,}26.$$

Since the Koch snowflake is a curve, we would expect its dimension to be 1. This discrepancy is due to the fact that the Koch snowflake is finally so fragmented that the resulting fractal has an infinite length but bounds a finite area plane structure.

Exercise 1. Calculate the perimeter of the Koch snowflake after the first, second, and third iterations.

Solution. Initially, we have an equilateral triangle with a perimeter $o_0 = 3$. In the first iteration, each of the three line segments is divided into thirds, and the middle third is replaced by two segments of length $\frac{1}{3}$. Each side of the original triangle is extended by $\frac{1}{3}$, resulting in a perimeter after the first iteration of

$$o_1 = 3 + 3 \cdot \frac{1}{3} = 4.$$

In the second iteration, each side of the original triangle is divided into four line-segments of one-third of the original length, which are further divided into thirds and extended by $\frac{1}{9}$. This leads to a perimeter of

$$o_2 = 3 + 3 \cdot \frac{1}{3} + 3 \cdot \frac{4}{9} = \frac{16}{3}.$$

In the third iteration, we extend 16 line-segments on each side by $\frac{1}{27}$, resulting in a perimeter of

$$o_3 = 3 + 3 \cdot \frac{1}{3} + 3 \cdot \frac{4}{9} + 3 \cdot \frac{16}{27} = 3 + 1 + \frac{4}{3} + \frac{16}{9} = \frac{64}{9}.$$

Exercise 2. What is the perimeter of the Koch snowflake after the n-th iteration? Prove that the perimeter of the Koch snowflake is infinite.

Solution. From the above calculations, we see that each line segment is one-third the length of the line segment from the previous iteration, and at the same time each segment is in the next iteration extended by one third, i.e., the segment is extended to $\frac{4}{3}$ of its previous length. The perimeter of the

Koch snowflake after the n-th iteration can be expressed using the sum of a geometric series with the common ratio of $\frac{4}{3}$ for $n \in \mathbb{N}$:

$$o_n = 3 + \left(\frac{4}{3}\right)^0 + \left(\frac{4}{3}\right)^1 + \left(\frac{4}{3}\right)^2 + \cdots + \left(\frac{4}{3}\right)^{n-1} = 3 + \sum_{i=1}^{n} \left(\frac{4}{3}\right)^{i-1}.$$

If we continued in this way indefinitely, we would obtain an infinite geometric series in the second term of the above sum. Since the ratio of the corresponding geometric sequence is greater than one, the series is divergent, and the perimeter of the Koch snowflake is infinite.

Exercise 3. Calculate the area of the Koch snowflake after the first and second iterations.

Solution. At the beginning, let us realize that the height of an equilateral triangle with side length a is $\frac{\sqrt{3}}{2}a$, and its area is given by

$$S = \frac{\sqrt{3}}{4}a^2.$$

The area of the initial equilateral triangle is $S_0 = \frac{\sqrt{3}}{4}$. In the first iteration, we divide the three line segments into thirds and we place a smaller equilateral triangle with side length $\frac{1}{3}$ in the middle third. The resulting area after the first iteration is

$$S_1 = \frac{\sqrt{3}}{4} + 3 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3}\right)^2 = \frac{\sqrt{3}}{4} \cdot \frac{4}{3}.$$

In the second iteration, each side of the original triangle is divided into four segments, and a smaller equilateral triangle with side length $\frac{1}{9}$ is placed on each segment. The area after the second iteration will increase to

$$S_2 = \frac{\sqrt{3}}{4} + 3 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3}\right)^2 + 3 \cdot 4 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{9}\right)^2 = \frac{\sqrt{3}}{4}\left(1 + \frac{1}{3} + \frac{1}{3} \cdot \frac{4}{9}\right) = \frac{\sqrt{3}}{4} \cdot \frac{40}{27}.$$

Exercise 4. What is the area of the Koch snowflake after the n-th iteration? How many times larger is the area of the Koch snowflake relative to the original equilateral triangle?

Solution. From the previous considerations, it follows that the number of segments, where we add a new triangle, is four times greater in each iteration. At the same time, the side of our new triangle shrinks to a third of its previous size, so its area decreases to one-ninth. We obtain terms of a geometric sequence with a ratio of $\frac{4}{9}$, and the area of the Koch snowflake after the n-th iteration is formed by the area of the original triangle and the sum of the first n terms of that geometric sequence:

$$S_n = \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3} + \frac{1}{3} \cdot \frac{4}{9} + \cdots + \frac{1}{3}\left(\frac{4}{9}\right)^{n-1}\right] = \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3}\sum_{i=1}^{n}\left(\frac{4}{9}\right)^{i-1}\right].$$

Since the ratio of the geometric sequence is less than one, by continuing to infinity, we obtain a convergent geometric series. Using the formula for its sum, we get the area of the Koch snowflake after an infinite number of iterations.

$$S = \frac{\sqrt{3}}{4}\left(1 + \frac{1}{3} \cdot \frac{1}{1 - \frac{4}{9}}\right) = \frac{\sqrt{3}}{4}\left(1 + \frac{1}{3} \cdot \frac{9}{5}\right) = \frac{8}{5} \cdot \frac{\sqrt{3}}{4} = 1{,}6 \cdot S_0.$$

The Koch snowflake has an infinite perimeter enclosing a finite area that is approximately 1.6 times larger than the area of the original equilateral triangle.

Literature

- MathWorld. *Koch snowflake* [online]. Available from https://mathworld.wolfram.com/KochSnowflake.html [cit. 13. 7. 2023].
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